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Hyers-Ulam Stability of Functional Equation that have Quadratic property

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ABSTRACT: The aim of this paper is to prove the stability of functional Equation that have Quadratic property in sprit of Hyers-Ulam.

Keywords : Hyers-Ulam Stability, Quadratic functional Equation
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I. INTRODUCTION

The problem of stability of homomorphism stemmed from the question posed by S.M. Ulam in 1940 in his lecture before the mathematical club of the University of Wisconsin. He demanded an answer to the following question of stability of homomorphism for metric groups.

Let G' be a group and let G'' be a metric group with the metric d. Given

$\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G' \to G''$ satisfies the following inequality

$d[h(xy), h(x)h(y)] < \delta$ for all x and y in G',

then there exists a homomorphism $H : G' \to G''$ with

$d[h(x), H(x)] < \varepsilon$ for all x in G' ?

In 1941 D.H. Hyers answered his question considering the case of Banach spaces. D.H. Hyers [2] proved the following result where E' and E'' are Banach spaces.

III. RESULT

Let $f : E' \to E''$ be a mapping between Banach spaces. If f satisfies the following inequality

$\| f(x + y) - f(x) - f(y) \| \leq \delta$

for all x and y in E' and some $\delta > 0$ then the limit

$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$

exists for all x in E' and $T : E' \to E''$ is a unique additive mapping such that

$\| f(x) - T(x) \| \leq \delta$ for all x in E'.

Moreover, if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.

The method adopted by D.H. Hyers is designated as *Direct Method* and the stability of any functional inequality which had an independent bound is termed as ***Hyers - Ulam Stability***.

F. Skof [7] was the first author who treated the Hyers - Ulam stability of the quadratic functional equation in the following manner.

Let $f : E' \to E''$ where E' is a normed space and E'' is a Banach space. If following inequality is satisfies

$\| f(x + y) + f(x - y) - 2f(x) + 2f(y) \| \leq \delta$

for all x and y in E' and for some $\delta > 0$, then for every x in E' the limit

$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$

exists and Q is the unique quadratic mapping which satisfies

$\| f(x) - Q(x) \| \leq \delta/2$ for all x in E'.

In this paper Hyers - Ulam stability of the following functional inequality

$\| f(x - y + z) + f(x) + f(y) + f(z) - f(y - x) - f(y - z) - f(z + x) \| \leq \theta$ is obtained. This inequality is reduces to quadratic functional inequalities

$\| 2f(x) + 2f(y) - f(x - y) - f(x + y) \| \leq 4\theta$

Whereupon the result 1.2 is used to prove the existence of a unique quadratic function $Q : X \to Y$, such that $f - Q$ is bounded on X.

II. PRELIMINARIES

A. Quadratic functional Equation

Let X and Y be a normed and a Banach space, respectively, if there is no specification. A mapping $f : X \to Y$ is called a *quadratic mapping* if f satisfies the following quadratic functional equation

$f(x + y) + f(x - y) = 2f(x) + 2f(y)$

for all x and y in X.

B. Hyers-Ulam Stability

Theorem. Assume that a function $f : X \to Y$ satisfies the following inequality

$$\| f(x - y + z) + f(x) + f(y) + f(z) - f(y - x) - f(y - z) - f(z + x) \| \le \theta \qquad \ldots(3.11)$$

for all x, y, z in X. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \le (4/3)\theta \qquad \ldots(3.12)$$

for all x in X . Moreover, if $f(tx)$ is continuous in t in R for each fixed x in X, then Q satisfies $Q(tx) = t^2 Q(x)$, for all x in X and t in R.

Proof: Put $x = y = z = 0$ in (3.11) to obtain

$$\| f(0) \| \le \theta$$

Now put $y = z = 0$ in (3.11) and then substitute y for x

$$\| f(y) - f(-y) \| \le 2\theta \qquad \text{for all } y \text{ in } X .$$

Now put $z = y$ in (3.11) to obtain

$$\| 2f(x) + 2f(y) - f(y - x) - f(y + x) \| \le 2\theta$$

or

$$\| 2f(x) - 2f(y) - f(x + y) - f(x - y) + f(x - y) - f(y - x) \| \le 2\theta$$

or

$$\| 2f(x) + 2f(y) - f(x + y) - f(x - y) \| \le 4\theta \qquad \ldots(3.13)$$

for all x and y in X , which is a quadratic equation.

Therefore according to result 1.2, it follows from the last inequality that there exists a unique quadratic function $Q : X \to Y$, and with the use of direct method the definition of Q is

$$Q(x) = \lim_{n\to\infty} \left(\frac{f(2^n x)}{2^{2n}} \right)$$

for all x in X.

From (3.13) it follows that

$$\| f(x) - Q(x) \| \le \sum_{n=0}^{\infty} (4\theta / 2^{2n+2})$$

$$\le (4\theta / 3)$$

This is exactly (3.12).

Uniqueness of Q

Let $T : X \to Y$ be another quadratic mapping that satisfies inequality (3.11) and (3.12). Obviously

$$Q(2^n x) = 4^n Q(x) \quad \text{and} \quad T(2^n x) = 4^n T(x)$$

for all x in X and n being positive. Therefore

$$\| Q(x) - T(x) \| = 4^{-n} \| Q(2^n x) - T(2^n x) \|$$
$$\le 4^{-n} \| Q(2^n x) - f(2^n x) \| + 4^{-n} \| T(2^n x) - f(2^n x) \| \le 4^{-n} (8\theta / 3)$$

for all x in X. By letting $n \to \infty$ in the preceding inequality, readily there is a uniqueness of Q.

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